

BOUNDING SECTIONAL CURVATURE ALONG A KÄHLER-RICCI FLOW

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ABSTRACT. If a normalized Kähler-Ricci flow $g(t), t \in [0, \infty)$, on a compact Kähler manifold M , $\dim_{\mathbb{C}} M = n \geq 3$, with positive first Chern class satisfies $g(t) \in 2\pi c_1(M)$ and has curvature operator uniformly bounded in L^n -norm, the curvature operator will also be uniformly bounded along the flow. Consequently the flow will converge along a subsequence to a Kähler-Ricci soliton.

1. INTRODUCTION

On a compact Kähler manifold M , $\dim_{\mathbb{C}} M = n$, with the first Chern class $c_1(M) > 0$, the normalized Kähler-Ricci flow equation is

$$(1.1) \quad \partial_t g(t) = -Ric(g(t)) + g(t),$$

for a family of Kähler metrics $g(t) \in 2\pi c_1(M)$, where, for brevity, $g(t)$ denotes either Kähler metrics or Kähler forms depending on the context. In [5], it is proved that a solution $g(t)$ of (1.1) exists for all $t \in [0, \infty)$. Perelman (cf. [20]) has proved some important properties for the solution $g(t)$, $t \in [0, \infty)$, of (1.1): there exist constants $C > 0$ and $\kappa > 0$ independent of t such that

- (1) $|R(g(t))| < C$, and $\text{diam}_{g(t)}(M) < C$,
- (2) $(M, g(t))$ is κ -noncollapsed, i.e. for any $r < 1$, if $|R(g(t))| \leq r^{-2}$ on a metric ball $B_{g(t)}(x, r)$, then

$$(1.2) \quad \text{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^{2n}.$$

In a recent preprint [19], Sesum has proved that, if $n \geq 3$, assuming the Ricci curvatures $|Ric(g(t))| < C$ and the integral of curvature operators $\int_M |Rm(g(t))|^n dv_t \leq C$, for a constant C independent of t , then the curvature operators are uniformly bounded. In this note, we will show that the hypothesis of bounded Ricci curvature can be removed.

Theorem 1.1. *Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler manifold M with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$. Assume that $\dim_{\mathbb{C}} M = n \geq 3$. If the L^n -norms of curvature operators are uniformly bounded by a constant C , i.e.*

$$\int_M |Rm(g(t))|^n dv_t \leq C,$$

then there exists a constant $0 < \bar{C} < \infty$ such that

$$\sup_{M \times [0, \infty)} |Rm(g(t))| \leq \bar{C}.$$

Consequently the flow will converge along a subsequence to a Kähler-Ricci soliton.

From this theorem, it is a direct consequence of Hamilton's compactness theorem (c.f. [11]) that, for any $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k + t))$, $t \in [0, 1]$, converges smoothly to $(X, h(t))$, $t \in [0, 1]$, where X is a compact complex manifold, and $\{h(t)\}$, $t \in [0, 1]$, is a family of Kähler metrics that satisfies the Kähler-Ricci flow equation. Furthermore, from the arguments in the proof of Theorem 12 in [20], $h(t)$, $t \in [0, 1]$, satisfies the Kähler-Ricci soliton equation, i.e. there is a holomorphic vector field v on X such that

$$Ric(h) - h = \mathcal{L}_v h.$$

In [19], Sesum conjectured that Theorem 1.1, as stated for $n \geq 3$, is also true for $n = 2$. By the classification theory of complex surface, the only compact Kähler surfaces with $c_1(M) > 0$ are diffeomorphic to $\mathbb{CP}^2 \# l \overline{\mathbb{CP}}^2$, $0 \leq l \leq 8$, and $\mathbb{CP}^1 \times \mathbb{CP}^1$. By [21], each of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2 \# l \overline{\mathbb{CP}}^2$, $3 \leq l \leq 8$ or $l = 0$, admits a Kähler-Einstein metric. In [6] and [13], it is shown that $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ admits a non-trivial Kähler-Ricci soliton metric. Wang and Zhu [24] showed the same result later for $\mathbb{CP}^2 \# 2 \overline{\mathbb{CP}}^2$. By [23], on a compact Kähler surface M with $c_1(M) > 0$, if the initial metric $g(0)$ is invariant under a one-parameter group obtained from a Kähler-Ricci soliton metric on M , the curvatures stay uniformly bounded along the flow. The only remaining case is when M is a complex surface diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2 \# 2 \overline{\mathbb{CP}}^2$, with an initial metric $g(0)$ without any symmetry. In [9], it is proved that the Kähler-Ricci flow on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ or $\mathbb{CP}^2 \# 2 \overline{\mathbb{CP}}^2$ converges to an orbifold in the Gromov-Hausdorff sense.

By using the method in the proof of Theorem 1.1, we can give a different proof of the convergence of the Kähler-Ricci flow on \mathbb{CP}^2 (Theorem 3.3), which is already implied by [23]. In a very recent preprint [7], Chen and Wang claimed that the bounding of curvatures along the Kähler-Ricci flow on a toric Fano surface M (including $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and $\mathbb{CP}^2 \# 2 \overline{\mathbb{CP}}^2$) could be proved by using the fact that M is a toric manifold.

There is also an analogy to Theorem 1.1 in the real Ricci flow case.

Theorem 1.2. *Let $g(t)$, $t \in [0, T)$, be a solution to the Ricci flow, normalized or not, on a closed odd dimensional manifold M with $T < \infty$. Suppose that*

$$\int_M |Rm(g(t))|^{n/2} dv_t \leq C, \quad \text{and} \quad |R(g(t))| \leq C$$

for a constant $C < \infty$ independent of t , where $n = \dim_{\mathbb{R}}(M)$. Then there is another constant $\bar{C} < \infty$ such that

$$\sup_{M \times [0, T)} |Rm(g(t))| \leq \bar{C},$$

and so the flow can be extended over T .

The organization of the paper is as follows: In §2, we prove Theorem 1.1. In §3, we give some remarks for Kähler-Ricci on Fano surfaces. Then we prove Theorem 1.2 in §4.

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2. PROOF OF THEOREM 1.1

Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler manifold M , $\dim_{\mathbb{C}} M = n$, with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$. Assume that $\dim_{\mathbb{C}} M = n \geq 3$, and that

$$\int_M |Rm(g(t))|^n dv_t \leq C,$$

for a constant C independent of t . Perelman (cf. [20]) has proved that there exist constants $C > 0$, $\kappa > 0$ independent of t such that

- (1) $|R(g(t))| < C$, and $\text{diam}_{g(t)}(M) < C$,
- (2) $(M, g(t))$ is κ -noncollapsed, i.e. for any $r < 1$, if $|R(g(t))| \leq r^{-2}$ on a metric ball $B_{g(t)}(x, r)$, then

$$(2.1) \quad \text{Vol}_{g(t)}(B_{g(t)}(x, r)) \geq \kappa r^{2n}.$$

The proof of Theorem 1.1 relies on the following theorem due to Gang Tian:

Theorem 2.1 (Theorem 2 in [22]). *Let (N, J, g) be a complete non-compact Ricci-flat Kähler manifold with $\dim_{\mathbb{C}} M = n \geq 3$,*

$$\int_N |Rm(g)|^n dv_g < C < \infty, \quad \text{and}$$

$$\text{Vol}_g(B_g(x, r)) \geq \kappa r^{2n},$$

for any $r > 0$, where C and κ are constants. Then M is a resolution of \mathbb{C}^n/Γ where Γ is a finite group $\Gamma \subset SU(n)$, which acts on $\mathbb{C}^n \setminus \{0\}$ freely, i.e. there is a holomorphic map $\pi : N \longrightarrow \mathbb{C}^n/\Gamma$ such that $\pi : N \setminus \pi^{-1}(0) \longrightarrow \mathbb{C}^n \setminus \{0\}/\Gamma$ is bi-holomorphic.

The assumptions of Euclidean volume growth and $\dim_{\mathbb{C}} M = n \geq 3$ not mentioned explicitly in Theorem 2 in [22] seem to be necessary. Let's recall several main steps in the proof of Theorem 2.1. First, an estimate for the decrease of the sectional curvature of g is obtained by assuming the Euclidean volume growth, bounded L^n -norm of curvature operator, and the Ricci-flat metric (See Lemma 4.1 in [22]). Then, for proving Theorem 2 of [22], one needs Lemma 3.4 and Lemma 3.3 of [22], which have the hypothesis $\dim_{\mathbb{C}} M = n \geq 3$. The main tool there was Kohn's estimate for \square_b -operators that works only for $n \geq 3$ (See [22] for details).

Proof of Theorem 1.1. Suppose otherwise, there exists a sequence of times $t_k \rightarrow \infty$, and a sequence of points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \rightarrow \infty.$$

Consider the sequence $\tilde{g}_k(t), t \in [-Q_k t_k, 0]$, where $\tilde{g}_k(t) = Q_k g(Q_k^{-1} t + t_k)$ and satisfy

$$(2.2) \quad \partial_t \tilde{g}_k(t) = -Ric(\tilde{g}_k(t)) + Q_k^{-1} \tilde{g}_k(t),$$

$$(2.3) \quad \sup_{M \times [-Q_k t_k, 0]} |Rm(\tilde{g}_k(t))| \leq 1, \quad \text{and} \quad |Rm(\tilde{g}_k(0))|(x_k) = 1.$$

By Perelman's estimate, we obtain that

$$(2.4) \quad |R(\tilde{g}_k(t))| < C Q_k^{-1} \rightarrow 0,$$

when $k \rightarrow \infty$, and, for any $r < C Q_k^{\frac{1}{2}}$, $x \in M$,

$$(2.5) \quad \text{Vol}_{\tilde{g}_k(t)}(B_{\tilde{g}_k(t)}(x, r)) \geq \kappa r^{2n}.$$

By Hamilton's compactness theorem (c.f. Appendix E in [12]), by passing to a subsequence, $\{(M, J, \tilde{g}_k(t), x_k)\}$, $t \in [-1, 0]$, converges smoothly to a family of pointed complete Riemannian manifold $(N, J_\infty, g_\infty(t), x_\infty)$, $t \in [-1, 0]$, where $g_\infty(t)$, $t \in [-1, 0]$, satisfies the Ricci-flow equation $\partial_t g_\infty(t) = -Ric(g_\infty(t))$. Particularly, for any $k \gg 1$ and $r > 0$, there is an embedding $F_{k,r} : B_{g_\infty}(x_\infty, r) \rightarrow M$ such that $F_{k,r}^* \tilde{g}_k(0)$ converges smoothly to g_∞ , and $dF_{k,r}^{-1} J dF_{k,r}$ converges smoothly to an almost complex structure J_∞ , where $g_\infty = g_\infty(0)$. Actually, J_∞ is integrable, and g_∞ is a Kähler metric of J_∞ (c.f. [17]). From (2.3) and (2.5), we obtain that

$$(2.6) \quad |Rm(g_\infty)| \leq |Rm(g_\infty)|(x_\infty) = 1,$$

and, for any $r > 0$ and $x \in N$,

$$(2.7) \quad \text{Vol}_{g_\infty(t)}(B_{g_\infty(t)}(x, r)) \geq \kappa r^{2n}.$$

By (2.4), $R(g_\infty(t)) \equiv 0$, which implies that $Ric(g_\infty(t)) \equiv 0$ since $g_\infty(t)$ is a solution of the Ricci-flow equation. From the smooth convergence,

$$\int_N |Rm(g_\infty)|^n dv_{g_\infty} \leq \limsup_{k \rightarrow \infty} \int_M |Rm(g(t_k))|^n dv_{g(t_k)} \leq C < \infty.$$

Thus (N, J_∞, g_∞) is a complete Ricci-flat Kähler manifold with Euclidean volume growth, and L^n -norm of curvature operator bounded. By (2.6), g_∞ is not a flat metric. Note that $\dim_{\mathbb{C}} M = n \geq 3$. By Theorem 2.1, N is a resolution of \mathbb{C}^n/Γ where Γ is a finite group $\Gamma \subset SU(n)$, which acts on $\mathbb{C}^n \setminus \{0\}$ freely, i.e. there is a holomorphic map $\pi : N \rightarrow \mathbb{C}^n/\Gamma$ such that $\pi : N \setminus \pi^{-1}(0) \rightarrow \mathbb{C}^n \setminus \{0\}/\Gamma$ is bi-holomorphic. If Γ is trivial, i.e. $\Gamma = \{e\}$, then (N, g_∞) is isometric to \mathbb{R}^{2n} by Theorem 3.5 in [1], which contradicts (2.1.5). Thus Γ is non-trivial, and $V = \pi^{-1}(0)$ is a compact analytic subvariety of (N, J_∞) with $0 < \dim_{\mathbb{C}} V = m < n$ (See [10] for the definition of $\dim_{\mathbb{C}} V$). We obtain that

$$\int_V g_\infty^m = m! \text{Vol}_{g_\infty}(V) > 0.$$

Note that, for any k , $F_{k,r}(V)$ is a cycle, and defines a homology class $[F_{k,r}(V)] \in H_{2m}(M, \mathbb{Z})$. By the smooth convergence of $F_{k,r}^* \tilde{g}_k$, for any $\varepsilon > 0$, there is a $k_0 > 0$ such that, for any $k \geq k_0$,

$$\left| \int_V g_\infty^m - \int_{F_{k,r}(V)} \tilde{g}_k^m \right| \leq \varepsilon,$$

where $\tilde{g}_k = \tilde{g}_k(0)$, and $r \gg 1$ such that $V \subset B_{g_\infty}(x_\infty, r)$. As $g_k = g(t_k) \in 2\pi c_1(M)$,

$$\int_{F_{k,r}(V)} \tilde{g}_k^m = Q_k^m \int_{F_{k,r}(V)} g_k^m = Q_k^m (2\pi)^m \int_{F_{k,r}(V)} c_1^m(M).$$

By taking $\varepsilon = \frac{1}{2} \int_V g_\infty^m$ and $k \gg k_0$, we obtain that

$$0 < \frac{1}{2} Q_k^{-m} (2\pi)^{-m} \int_V g_\infty^m \leq \int_{F_{k,r}(V)} c_1^m(M) \leq \frac{3}{2} Q_k^{-m} (2\pi)^{-m} \int_V g_\infty^m < 1.$$

Since $0 \neq c_1^m(M) \in H^{2m}(M, \mathbb{Z})$, and $[F_{k,r}(V)] \in H_{2m}(M, \mathbb{Z})$, we have

$$\int_{F_{k,r}(V)} c_1^m(M) \in \mathbb{Z}.$$

It is a contradiction. We obtain that

$$\sup_{M \times [0, \infty)} |Rm(g(t))| \leq \bar{C},$$

for a constant $\bar{C} > 0$.

Now, by Hamilton's compactness theorem (c.f. [11]), for any $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k + t))$, $t \in [0, 1]$, converges smoothly to a family of compact Kähler manifolds $(X, h(t))$, $t \in [0, 1]$, where $h(t)$ satisfies the Kähler-Ricci flow equation. Actually, $h(t)$, $t \in [0, 1]$, satisfies the Kähler-Ricci soliton equation from the arguments in the proof of Theorem 12 in [20].

□

3. REMARKS FOR KÄHLER SURFACES

Let $g(t)$, $t \in [0, +\infty)$, be a solution of the normalized Kähler-Ricci flow (1.1) on a compact Kähler surface M , i.e. $\dim_{\mathbb{C}} M = 2$, with $c_1(M) > 0$ and initial metric $g(0) \in 2\pi c_1(M)$.

Lemma 3.1. *The L^2 -norms of curvature operators of $g(t)$ are bounded along the flow, i.e. there is a constant $C > 0$ independent of t such that*

$$(3.1) \quad \int_M |Rm(g(t))|^2 dv_t \leq C.$$

Proof. Since, for any $t \in [0, \infty)$, $(M, g(t))$ is a Kähler surface, we have $\int_M c_1^2(M) = 2\chi(M) + 3\tau(M)$, $R^2(g(t)) = 24|W^+(g(t))|^2$, and Gauss-Bonnet-Chern formula and Hirzebruch formula

$$\begin{aligned}\chi(M) &= \frac{1}{8\pi^2} \int_M \left(\frac{R^2}{24} + |W^+|^2 + |W^-|^2 - \frac{1}{2}|Ric^o|^2 \right) dv_t, \\ \tau(M) &= \frac{1}{12\pi^2} \int_M (|W^+|^2 - |W^-|^2) dv_t\end{aligned}$$

(c.f. [2]), where $Ric^o = Ric(g(t)) - \frac{R}{4}g(t)$, $W^\pm(g(t))$ are the self-dual and anti-self-dual Weyl tensors of $g(t)$, and $\chi(M)$ (respectively $\tau(M)$) is the Euler number (respectively signature) of M . Then we obtain that

$$\begin{aligned}\int_M |Ric^o|^2 dv_t &= \int_M \frac{R^2}{4} dv_t - 8\pi^2 c_1^2(M), \quad \text{and} \\ (3.2) \quad \int_M |W^-|^2 dv_t &= \int_M \frac{R^2}{24} dv_t - 12\pi^2 \tau(M).\end{aligned}$$

Note that

$$Rm(g(t)) = \begin{pmatrix} W^+ + \frac{R}{12} & Ric^o \\ Ric^o & W^- + \frac{R}{12} \end{pmatrix}.$$

Thus we obtain that

$$\begin{aligned}\int_M |Rm(g(t))|^2 dv_t &= \int_M \left(\frac{R^2}{24} + |W^+|^2 + |W^-|^2 + 2|Ric^o|^2 \right) dv_t \\ &= \int_M \frac{5R^2}{8} dv_t - 16\pi^2 c_1^2(M) - 12\pi^2 \tau(M).\end{aligned}$$

Hence, by Perelman's estimate for scalar curvatures, we obtain (3.1). \square

Unfortunately, our arguments in the proof of Theorem 1.1 can not be generalized to this case, even for $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$. The essential point in the proof of Theorem 1.1 is that, in any Asymptotically Locally Euclidean Ricci-flat Kähler manifold N of $\dim_{\mathbb{C}} N \geq 3$, we can find a non-trivial class $[V] \neq 0 \in H_{2m}(N, \mathbb{Z})$ with $m \geq 1$. However, there are ALE Ricci-flat Kähler surfaces without such homology classes. For example, there is an ALE Ricci-flat Kähler metric h on $T^*\mathbb{RP}^2$ whose Betti numbers satisfy $b_2 = b_3 = b_4 = 0$. Actually, the universal covering space of $T^*\mathbb{RP}^2$ with the pull-back metric is the Eguchi-Hanson space (c.f. [8]), which is diffeomorphic to T^*S^2 .

Proposition 3.2. *Assume that M is diffeomorphic to $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$. If there is a sequence of times $t_k \rightarrow \infty$ such that*

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \rightarrow \infty,$$

where $x_k \in M$, then a subsequence of $(M, Q_k g(t_k), x_k)$ converges smoothly to an ALE Ricci-flat Kähler surface (N, g_∞, x_∞) in the pointed Gromov-Hausdorff sense. Furthermore, the fundamental group $\pi_1(N)$ of N is a non-trivial finite group.

Proof. Let $\tilde{g}_k = Q_k g(t_k)$. By the same arguments as in the proof of Theorem 1.1, by passing to a subsequence, $\{(M, J, \tilde{g}_k, x_k)\}$ converges smoothly to a complete Ricci-flat Kähler surface $(N, J_\infty, g_\infty, x_\infty)$, i.e. for any $k \gg 1$ and $r > 0$, there is an embedding $F_{k,r} : B_{g_\infty}(x_\infty, r) \rightarrow M$ such that $F_{k,r}^* \tilde{g}_k$ converges smoothly to g_∞ , and $dF_{k,r}^{-1} J dF_{k,r}$ converges smoothly to J_∞ . Furthermore, (N, J_∞, g_∞) satisfies that, for any $r > 0$ and $x \in N$,

$$\text{Vol}_{g_\infty}(B_{g_\infty}(x, r)) \geq \kappa r^4,$$

$$\int_N |Rm(g_\infty)|^2 dv_{g_\infty} \leq C < \infty,$$

$$\text{and } |Rm(g_\infty)| \leq |Rm(g_\infty)|(x_\infty) = 1.$$

Thus, by Theorem 1.5 in [3], (N, J_∞, g_∞) is an Asymptotically Locally Euclidean Ricci-flat Kähler surface. Since g_∞ is not flat, it is easy to see that the fundamental group $\pi_1(N)$ of N is a finite group (c.f. [1]).

If $\pi_1(N) = \{1\}$, (N, J_∞, g_∞) is an ALE hyper-Kähler 4-manifold. By the classification theory of ALE hyper-Kähler 4-manifold (c.f. [14]), there is a close surface $\Sigma \subset N$ such that $[\Sigma] \in H_2(N, \mathbb{Z})$, and $[\Sigma] \cdot [\Sigma] = -2$. Then, for $k \gg 1$ and $r \gg 1$, $F_{k,r}(\Sigma)$ is a cycle in M , and defines a homology class $[F_{k,r}(\Sigma)] \in H_2(M, \mathbb{Z})$ with $[F_{k,r}(\Sigma)] \cdot [F_{k,r}(\Sigma)] = -2$. Let $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$ such that $[F_{k,r}(\Sigma)] = aH + bE$, where H and E are the two generators of $H_2(M, \mathbb{Z})$ such that $H \cdot H = 1$, $E \cdot E = -1$ and $H \cdot E = 0$. However, the equation $a^2 - b^2 = [F_{k,r}(\Sigma)] \cdot [F_{k,r}(\Sigma)] = -2$ does not have integer solutions. It is a contradiction. Thus $\pi_1(N) \neq \{1\}$. \square

Actually, by the same arguments as in the proof of (2) in Theorem 5.2 of [21], we can see that $\pi_1(N)$ is a cyclic finite group, and (N, J_∞, g_∞) is asymptotic to \mathbb{C}^2/Γ , where Γ is a finite cyclic subgroup of $U(2)$ given by Lemma 5.5 in [21]. If one wants to use the technique in the proof of Theorem 1.1 to prove the bounding of curvatures along the Kähler-Ricci flow on $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$, a method must be found to prove that N is actually simply connected. In a recently preprint [7], it is claimed that this could be done by using the fact that $\mathbb{CP}^2 \sharp \overline{\mathbb{CP}^2}$ is a toric manifold. However, the full details of the arguments in [7] have not appeared yet.

Our method can be used to give a different proof of the following theorem in term of Gromov-Hausdorff convergence, which is already implied by [23] where the Monge-Ampère flow was used.

Theorem 3.3. ([23]) *If M is holomorphic to \mathbb{CP}^2 , and $g(t)$, $t \in [0, +\infty)$, is a solution of the normalized Kähler-Ricci flow (1.1), then, for any sequence of times $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k))$ converges smoothly to the unique Kähler-Einstein metric on \mathbb{CP}^2 in the Cheeger-Gromov sense.*

Proof. It is well known that, on \mathbb{CP}^2 , there is a unique Kähler-Einstein metric presenting $2\pi c_1$, the Fubini-Study metric (c.f. [4]). This implies that the Mabuchi's K-energy $\nu_{g_0}(g(t))$ is bounded from below (c.f. [16]). Thus, by Perelman's estimate for scalar

curvatures, (6.1) in [16] holds, i.e.

$$\int_M |\partial\bar{\partial}u_t|^2 dv_t \longrightarrow 0,$$

when $t \rightarrow \infty$, where u_t are functions satisfying $-Ric(g(t)) + g(t) = \sqrt{-1}\partial\bar{\partial}u_t$. Since the Hodge Laplacian satisfies $\Delta = 2(\partial^*\partial + \partial\partial^*) = 2(\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*)$, $\Delta\bar{\partial} = \bar{\partial}\Delta$, and $\Delta u_t = R(g(t)) - 4$, we obtain that

$$(3.3) \quad \int_M |R(g(t)) - 4|^2 dv_t = \int_M |\Delta u_t|^2 dv_t = 4 \int_M |\partial\bar{\partial}u_t|^2 dv_t \longrightarrow 0.$$

Then

$$\begin{aligned} \lim_{t \rightarrow \infty} \int_M R(g(t))^2 dv_t &= \lim_{t \rightarrow \infty} \int_M (8R(g(t)) - 16) dv_t \\ &= \lim_{t \rightarrow \infty} (16 \int_M Ric(g(t)) \wedge g(t) - 8 \int_M g(t) \wedge g(t)) \\ &= 32\pi^2 \int_M c_1^2(M) = 32\pi^2(2\chi(M) + 3\tau(M)). \end{aligned}$$

By (3.2), we have

$$\lim_{t \rightarrow \infty} \int_M |W^-(g(t))|^2 dv_t = \frac{8}{3}\pi^2(\chi(M) - 3\tau(M)) = 0,$$

since $\chi(M) = 3$ and $\tau(M) = 1$.

If $\sup_{t \in [0, \infty)} |Rm(g(t))| = +\infty$, then there exists a sequence of times $t_k \rightarrow \infty$, and a sequence of points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \rightarrow \infty.$$

Let $\tilde{g}_k = Q_k g(t_k)$. By the same arguments as in the proof of Theorem 1.1, by passing to a subsequence, $\{(M, J, \tilde{g}_k, x_k)\}$ converges smoothly to a complete Ricci-flat Kähler surface $(N, J_\infty, g_\infty, x_\infty)$ with

$$\sup_N |Rm(g_\infty)| = 1.$$

Furthermore, by the smooth convergence,

$$\int_N |W^-(g_\infty)|^2 dv_\infty \leq \lim_{t \rightarrow \infty} \int_M |W^-(g(t))|^2 dv_t = 0, \quad \text{thus } W^-(g_\infty) \equiv 0,$$

on N . From $R^2(g_\infty) = 24|W^+(g_\infty)|^2 \equiv 0$, we obtain that $|Rm(g_\infty)| \equiv 0$ on N . It is a contradiction. Hence there is a constant $C > 0$ independent of t such that

$$|Rm(g(t))| \leq C.$$

Finally, by the same arguments as in the proof of Theorem 1.1, for any $t_k \rightarrow \infty$, a subsequence of $(M, g(t_k + t))$, $t \in [0, 1]$, converges smoothly to a compact Kähler-Ricci soliton $(X, h(t))$, $t \in [0, 1]$. By (3.3), $R(h(t)) \equiv 4$, and, thus, $h(0)$ is a Kähler-Einstein metric. By Kodaria classification theorem, it is well known that the Fano surface diffeomorphic to \mathbb{CP}^2 is unique. Therefore, $X \cong M$. \square

4. PROOF OF THEOREM 1.2

The proof of Theorem 1.2 follows by a similar argument as in the proof of Theorem 1.1.

Proof of Theorem 1.2. Suppose not, there exist a sequence of $t_k \rightarrow T$ and points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \rightarrow \infty.$$

Then consider the sequence of solutions to the Ricci flow

$$(M, Q_k g(Q_k^{-1}t + t_k), x_k), t \in [-Q_k t_k, 0].$$

First we assume $g(t)$ is a solution to the unnormalized Ricci flow. Perelman's no local collapsing theorem (cf. [15, Theorem 4.1] or [12, Remark 12.13]) applies to show that there is a $\kappa > 0$ such that $Vol(B(x, r, g(t))) \geq \kappa r^n$ for each metric ball $B(x, r, g(t))$ in $(M, g(t))$ with radius $r \leq \sqrt{T}$. Then using Hamilton's compactness theorem for Ricci flow solutions, the sequence will converge modulo a subsequence to another solution to the Ricci flow, say $(M_\infty, g_\infty(t), x_\infty)$, $t \in (-\infty, 0]$, which has the properties that $Vol(B(r, g_\infty(t))) \geq \kappa r^n$, for each metric ball $B(r, g_\infty(t))$ in $(M_\infty, g_\infty(t))$ of radius r , and that

$$\begin{aligned} \int_{M_\infty} |Rm(g_\infty(t))|^{n/2} dv_{g_\infty(t)} &\leq \limsup_{k \rightarrow \infty} \int_M |Rm(g(t))|^{n/2} dv_t < \infty, \\ |Rm(g_\infty(0))|(x_\infty) &= 1 \text{ and } R(g_\infty(t)) \equiv 0 \text{ over } M_\infty \times (-\infty, 0]. \end{aligned}$$

From the evolution of the volume $Vol(g(t))$ of the metric $g(t)$:

$$\frac{d}{dt} Vol(g(t)) = - \int_M R(g(t)) dv_{g(t)} \geq -C Vol(g(t)),$$

we conclude that $Vol(g(t)) \geq Vol(g(0))e^{-Ct} \geq Vol(g(0))e^{-CT}$ for each metric $g(t)$. So the limits $(M_\infty, g_\infty(t))$ are non-compact Ricci flat manifolds. After a double covering, we may also assume that the manifold M is oriented and so the limit M_∞ is also oriented. By odd dimensional assumption of M , using theorem 3.5 of [1], we conclude that $(M_\infty, g_\infty(0))$ is in fact the Euclidean space, which contradicts the fact $|Rm(g_\infty(0))|(x_\infty) = 1$.

If $g(t)$ is a solution to the normalized Ricci flow, then the rescaling factor from $g(t)$ to the corresponding unnormalized Ricci flow is uniformly bounded from above and below (stays bounded away from zero), since the scalar curvature of $g(t)$ is absolutely bounded. Thus the corresponding unnormalized Ricci flow exists in finite time, and then Perelman's no local collapsing theorem uses also. So repeatedly, $Vol(B(x, r, g(t))) \geq \kappa r^n$ for each metric ball $B(x, r, g(t))$ in $(M, g(t))$ with radius $r \leq \sqrt{T}$, for some universal

$\kappa > 0$. If $g(t)$ does not have uniformly bounded Riemannian curvature, then there is a sequence of times $t_k \rightarrow T$ and points $x_k \in M$ such that

$$Q_k = |Rm(g(t_k))|(x_k) = \sup_{M \times [0, t_k]} |Rm(g(t))| \rightarrow \infty.$$

Then consider the sequence of solutions to the Ricci flow

$$(M, Q_k g(Q_k^{-1}t + t_k), x_k), t \in [-Q_k t_k, 0],$$

which will converge along a subsequence to a Ricci flat solution on an open manifold. The limit solution is flat by a same argument and we obtain a contradiction. \square

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